Homework 5

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Problem 2.5.3

This was done on the previous homework.

Problem 2.6.1

a) An example of a Cauchy sequence that is not monotone is: $a_n = \frac{(-1)^n}{n}$

b) An example of a monotone sequence that is not Cauchy is: $a_n = n$

c) No Cauchy sequence can contain a divergent subsequence because by **Theorem 2.5.2**: subsequences of convergent sequences converge to the same limit as the original sequence, which means they must converge.

d) An example of an unbounded sequence that contains a Cauchy subsequence is: $a_n = n\sin(\frac{(n\pi)}{2})$. The Cauchy subsequence is $b_n = a_{2n}$.

Problem 2.6.3

a) Pseudo-Cauchy is a specific case of Cauchy where an index can be chosen for an arbitrary $\epsilon > 0$ so that the distance between each term and the term following it after this index is less than this ϵ. This is looser definition of Cauchy because Cauchy necessitates that the distance between *any* two terms after that index is less than this arbitrary ϵ.

b) $a_n = \sum_1^n \frac{1}{n}$ is an example of a divergent sequence that is Pseudo-Cauchy because we know from the p-series test that $\sum_1^n \frac{1}{n}$ diverges and $\forall \epsilon > 0$ we can find an N such that every term after that N is arbitrarily close to its neighbor. Explicitly, for a given ϵ, we choose N such that $\epsilon > \frac{1}{N}$ (this is possible by the Archimedean Property).

Problem 2.6.4

Proof. We know the following:

$$\forall \epsilon_1 > 0 \; \exists N_1 \in \mathbb{N} | \forall n, m \geq N_1 \; |a_n - a_m| < \epsilon_1$$

$$\forall \epsilon_2 > 0 \; \exists N_2 \in \mathbb{N} | \forall n, m \geq N_2 \; |b_n - b_m| < \epsilon_2$$

Let ϵ_1 and ϵ_2 be equal to $\frac{\epsilon}{2}$.

We can take $N = max\{N_1(\epsilon_1), N_2(\epsilon_2)\}$.

From this, we can say that $\forall n, m \geq N \; |a_n - a_m| + |b_n - b_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ from that we know $|(a_n - a_m) - (b_n - b_m)| \leq |a_n - a_m| + |b_n - b_m| \leq < \epsilon$ and from that we know $||a_n - b_n| - |a_m - b_m|| \leq |a_n - b_n - a_m + b_n| \leq |(a_n - a_m) - (b_n - b_m)| < \epsilon$ from which we know $||a_n - b_n| - |a_m - b_m| < \epsilon \quad \forall n, m > 0$ which implies that $(a_n - b_n)$ is Cauchy. ☐

Problem 2.6.5.a

Proof. We start with the assumptions:

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} | \forall n, m \geq N \; |x_n - x_m| < \epsilon$$

and

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} | \forall n, m \geq N \; |y_n - y_m| < \epsilon$$

We want to end with:

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} | \forall n, m \geq N \; |x_n - x_m| + |y_n - y_m| < \epsilon$$

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Note to grader: I need help with learning how to format my latex documents correctly (I just started teaching myself a few days ago); is there anywhere you can direct me?